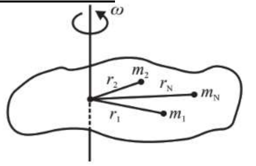
ROTATIONAL MOTION PART 2

Rotational KE: is the sum of translational KE of all individual particles



Let the object consists of N particles of mass $m_1, m_2, \ldots m_N$ at respective perpendicular distance of $r_1, r_2, \ldots, r_N$ from the axis of rotation. As the rigid body is rotating, all these particles are performing UCM with constant angular speed ω about an axis perpendicular to the plane of paper. But they all have different linear speeds $v_1=r_1\omega$, $v_2=r_2\omega$...... , $v_N=r_N\omega$

Translational KE is $KE_1= ½\, m_1v_1^2= ½\, m_1r_1^2\omega^2$, $KE_2= ½\, m_2v_2^2= ½\, m_2r_2^2\omega^2$,....

Rotational KE= $½\, m_1r_1^2\omega^2+ ½\, m_2r_2^2\omega^2+.......+ ½\, m_Nr_N^2\omega^2$

$= ½\,(m_1r_1^2+ m_2r_2^2+....+ m_Nr_N^2)\omega = ½\, I\omega^2$

where $I = m_1r_1^2+ m_2r_2^2+....+ m_Nr_N^2 = \sum_{i=1}^{N} m_i {r_i}^2$ is the rotational inertia or moment of Inertia (MI) of the object about the given axis of rotation.

NOTE: MI depends on individual masses and the distribution of these masses about the axis of rotation.

Angular Momentum in terms of MI:

Let the object consists of N particles of mass $m_1, m_2, \ldots m_N$ at respective perpendicular distance of $r_1, r_2, \ldots, r_N$ from the axis of rotation. As the rigid body is rotating, all these particles are performing UCM with constant angular speed ω about an axis perpendicular to the plane of paper. But they all have different linear speeds $v_1=r_1\omega$, $v_2=r_2\omega$...... , $v_N=r_N\omega$. These velocities are along the tangent.

Linear momentum is $p_1=m_1v_1=m_1r_1\omega$, $p_2=m_2v_2=m_2r_2\omega$,........

Angular momentum is $L_1= m_1r_1^2\omega$, $L_2= m_2r_2^2\omega$,....., $L_N= m_Nr_N^2\omega$

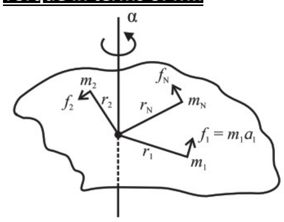
Since all have same direction, the magnitude of angular momentum is

$L= m_1r_1^2\omega + m_2r_2^2\omega +.....+ m_Nr_N^2\omega=(m_1r_1^2 + m_2r_2^2 +.....+ m_Nr_N^2)\omega = I\omega$

where $I = m_1r_1^2+ m_2r_2^2+....+ m_Nr_N^2 = \sum_{i=1}^{N} m_i {r_i}^2$ is the moment of Inertia (MI) of the object about the given axis of rotation.

NOTE: $L=I\omega$ is analogous to linear momentum $p=mv$, if moment of Inertia (I) replaces mass, which is its physical significance.

Torque in terms of MI:



Let the object consists of N particles of mass $m_1, m_2, \ldots m_N$ at respective perpendicular distance of $r_1, r_2, \ldots, r_N$ from the axis of rotation. As the rigid body is rotating, all these particles are performing CM with constant angular acceleration α about an axis perpendicular to the plane of paper. But they all have different linear(tangential) accelerations $a_1=r_1\alpha$, $a_2=r_2\alpha$ ...... , $a_N=r_N\alpha$. The force experienced by each particle is $f_1=m_1a_1=m_1r_1\alpha$, $f_2=m_2a_2=m_2r_2\alpha$,..... , $f_N=m_Na_N=m_Nr_N\alpha$

These forces are tangential and their respective perpendicular distances are r_1, r_2,...., r_N

Torque experienced is $\tau_1=f_1r_1= m_1r_1^2\alpha$, $\tau_2=f_2r_2= m_2r_2^2\alpha$,......., $\tau_N=f_Nr_N= m_Nr_N^2\alpha$

Magnitude of the resulting torque $\tau = \tau_1 + \tau_2 +...... +\tau_N$

$\tau = m_1r_1^2\alpha + m_2r_2^2\alpha + ... + m_Nr_N^2\alpha = (m_1r_1^2 + m_2r_2^2 + ... + m_Nr_N^2)\alpha$

$\tau=I\alpha$, where $I = m_1r_1^2+ m_2r_2^2+....+ m_Nr_N^2 = \sum_{i=1}^{N} m_i {r_i}^2$ is the moment of Inertia (MI) of the object about the given axis of rotation.

NOTE: $\tau=I\alpha$ is analogous to $f=ma$ of linear motion, if moment of Inertia (I) replaces mass, which is its physical significance.

Conservation of Angular Momentum:

Angular momentum $\bar{L} = \bar{r}\; X\; \bar{p}$ where $\bar{r}$ is the position vector from the axis of rotation and $\bar{p}$ is the linear momentum

Differentiating w.r.t. time, we get

$$\frac{d\bar{L}}{dt} = \frac{d}{dt}(\bar{r}\; X\; \bar{p}) = \bar{r}\; X\; \frac{d\bar{p}}{dt} + \frac{d\bar{r}}{dt} X \bar{p}$$

$$\frac{d\bar{L}}{dt} = \bar{r}\; X\; \bar{F} + m\,(\bar{v}\; X\; \bar{v}), Since\ \frac{d\bar{p}}{dt} = \bar{F}\ \ and\ \frac{d\bar{r}}{dt} = \bar{v}$$

$$Since\ \bar{v}\; X\; \bar{v} = \bar{0}\ ,$$

$$Thus\ \frac{d\bar{L}}{dt} = \bar{r}\; X\; \bar{F}\ which\ is\ moment\ of\ force\ of\ torque$$

$$\tau = \frac{d\bar{L}}{dt} \ldots\ldots If\ \tau = 0, \frac{d\bar{L}}{dt} = 0\ , hence\ \bar{L} = constant\ (conserved)$$

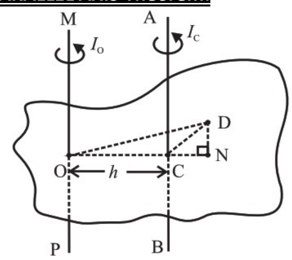
Hence in the absence of external unbalanced torque, $\bar{L}$ is conserved

Examples

Ballet Dancers: During ice ballets the dancers come together (MI decreases, frequency increases) while taking rounds of small radius. While taking outer rounds the dancers outstretch their legs and arms. This will increase the MI and reduces the angular speed and linear speed. This is necessary to prevent slipping

Diving into a swimming pool during a competition: When standing on a diving board the diver stretches to have higher MI. Immediately on leaving the board the diver folds his body, to decrease MI, increase his frequency so he can complete more rounds in air. While just entering the pool diver stretches the body into a streamline shape for smooth entry into water.

PARALLEL AXIS Theorem:



C be the center of mass

M Object mass

Axis of rotation MOP is parallel to axis ACB which is passing through center of mass and perpendicular to the plane.

h is the distance between the parallel axes.

Consider a mass dm at D. The MI about the two axes is $I_c = \int (DC)^2\, dm\ \ and\ I_O = \int (DO)^2\, dm$

$$I_O = \int [(DN)^2 + (ON)^2]\, dm\ \ But\ ON = OC + CN$$

$$I_O = \int [(DN)^2 + (OC + CN)^2]\, dm$$

$$= \int [(DN)^2 + (CN)^2 + (OC)^2 + 2(OC)(CN)]\, dm$$

$$I_O = \int [(DC)^2 + h^2 + 2(h)(CN)]\, dm$$

$$= \int (DC)^2 dm + h^2 \int dm + 2h \int CNdm$$

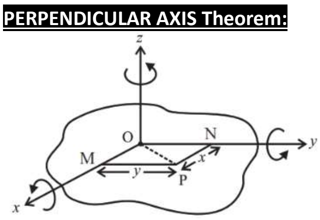
$$\int (DC)^2 dm = I_c,\ \int dm = M,\ \int CNdm = 0,$$

$$since\ mass\ distribution\ is\ symmetric$$

$$\boldsymbol{Therfore, I_o = I_c + Mh^2}$$

The MI (I_o) of an object about any axis is given by the sum of the moment of inertial about a parallel axis through the center of mass and product of the mass of the object and the square of the distance between the two axes (Mh^2).

PERPENDICULAR AXIS Theorem:

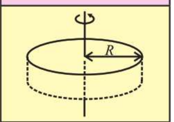
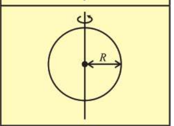
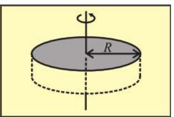
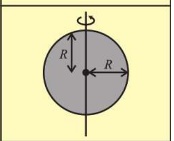
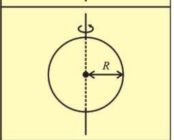
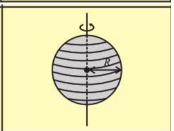
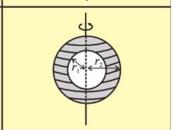
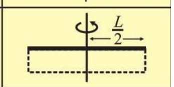
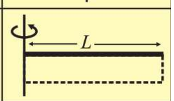
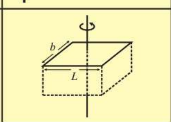
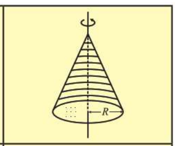
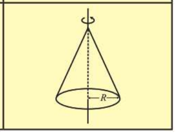


Let us consider a rigid laminar object able to rotate about three mutually perpendicular axes x,y and z. Let x and y axis be in the plane and z axis perpendicular to the plane. All 3 axes meet at O. Let dm be the mass at point P. x and y are the perpendicular distance of P from the y and x axis respectively. The distance from Z axis will be $\sqrt{x^2 + y^2}$. Let I_x, I_y and I_z be the MI of the body about the respective axes.

$$\boldsymbol{I_Z} = \int z^2 dm = \int (x^2 + y^2) dm = \int x^2 dm + \int y^2 dm = \boldsymbol{I_x + I_y}$$

The MI (I_z) of a laminar object about an axis (z) perpendicular to its plane is the sum of its MI about two mutually perpendicular axes (x and y) in its plane, all three axes being concurrent.

Equation for translational motion	Analogous equation for rotational motion
$v_{av} = \frac{u+v}{2}$	$\omega_{av} = \frac{\omega_0+\omega}{2}$
$a = \frac{dv}{dt} = \frac{v-u}{t}$ $\therefore v = u + at$	$\alpha = \frac{d\omega}{dt} = \frac{\omega-\omega_0}{t}$ $\therefore \omega = \omega_0 + \alpha t$
$s = v_{av} \cdot t$ $= \left(\frac{u+v}{2}\right)t$ $= ut + \frac{1}{2}at^2$	$\theta = \omega_{av} \cdot t$ $= \left(\frac{\omega_0+\omega}{2}\right)t$ $= \omega_0 t + \alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

Object	Axis	Expression of moment of inertia	Figure
Thin ring or hollow cylinder	Central	$I = MR^2$	
Thin ring	Diameter	$I = \frac{1}{2}MR^2$	
Annular ring or thick walled hollow cylinder	Central	$I = \frac{1}{2}M\left(r_2^2 + r_1^2\right)$	
Uniform disc or solid cylinder	Central	$I = \frac{1}{2}MR^2$	
Uniform disc	Diameter	$I = \frac{1}{4}MR^2$	
Thin walled hollow sphere	Central	$I = \frac{2}{3}MR^2$	
Solid sphere	Central	$I = \frac{2}{5}MR^2$	
Uniform symmetric spherical shell	Central	$I = \frac{2}{5}M\frac{\left(r_2^5 - r_1^5\right)}{\left(r_2^3 - r_1^3\right)}$	
Thin uniform rod or rectangular plate	Perpendicular to length and passing through centre	$I = \frac{1}{12}ML^2$	
Thin uniform rod or rectangular plate	Perpendicular to length and about one end	$I = \frac{1}{3}MR^2$	
Uniform plate or rectangular parallelepiped	Central	$I = \frac{1}{12}M(L^2 + b^2)$	
Uniform solid right circular cone	Central	$I = \frac{3}{10}MR^2$	
Uniform hollow right circular cone	Central	$I = \frac{1}{2}MR^2$	

Radius of Gyration:

$I = MK^2$ where K is called radius of gyration

Moment of inertia of an object of mass M can be represented by a point mass of value M (same as that as the object) and placed at a distance K from the axis of rotation so as to produce the same moment of inertia. K is called the radius of gyration $K = \sqrt{\frac{I}{M}}$.

Larger the value of K, more away is the mass from axis of rotation.

Rolling Motion:

In case of pure rolling motion,

Total KE= Translational KE of center of mass + Rotational KE

$$= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}(MK^2)\left(\frac{v}{R}\right)^2 = \frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right)$$

Where K = radius of gyration, R= radius of the body, v = linear speed of center of mass, ω=Angular speed of the body, I=Moment of inertia, M=mass of the body.

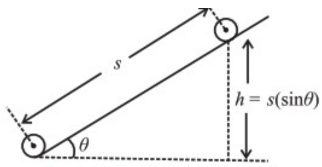
NOTE: $\text{Translational : Rotational : Total KE} = 1 : \frac{K^2}{R^2} : 1 + \frac{K^2}{R^2}$

Body rolling along an incline:

As the body rolls down,

TE on top = TE at the bottom

PE top = KE at bottom



$$mgh = \frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right)$$

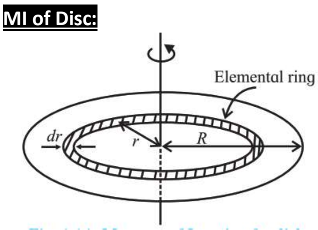
$$therefore, \boldsymbol{v} = \sqrt{\frac{\boldsymbol{2gh}}{\left(\boldsymbol{1} + \frac{\boldsymbol{K^2}}{\boldsymbol{R^2}}\right)}}$$

$$v^2 = u^2 + 2as,\ \ Thus\ a = \frac{v^2 - u^2}{2s}$$

$$Assuming\ body\ starts\ at\ rest, a = \frac{\frac{2gh}{\left(1 + \frac{K^2}{R^2}\right)}}{2\frac{h}{sin\theta}} = \frac{\boldsymbol{gsin\theta}}{\left(\boldsymbol{1} + \frac{\boldsymbol{K^2}}{\boldsymbol{R^2}}\right)}$$

NOTE: K^2/R^2 for ring=1 , for disc or solid cylinder= ½ , for solid sphere=2/5

MI of Disc:



Consider a disc of mass M and negligible thickness and radius R. Let the axis of rotation pass through the center of the disc and perpendicular to it place.

$Let\ \sigma = \frac{M}{A} = \frac{M}{\pi R^2}$ be uniform. A disc can be considered a sequence of rings. Consider one such ring at distance r and mass dm and negligible thickness dr. Then $\sigma = \frac{dm}{2\pi r dr}$. MI of this ring is I_r= dm.r²

The MI of the disc will be got by integrating this from r=0 to r=R

$$I = \int_o^R dm\, r^2 = \int_o^R \sigma 2\pi r dr.\, r^2 = 2\pi\sigma\int_o^R r^3 dr = 2\pi\sigma\left[\frac{r^4}{4}\right]_0^R$$

$$= 2\pi\frac{M}{\pi R^2}\frac{R^4}{4} = \frac{MR^2}{2}$$